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A THEORY OF CONDITIONAL INFORMATION FOR PROBABILISTIC INFERENCE IN INTELLIGENT SYSTEMS: III, MATHEMATICAL APPENDIX

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A THEORY OF CONDITIONAL INFORMATION
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IN INTELLIGENT SYSTEMS:
III, MATHEMATICAL APPENDIX

ABSTRACT

This paper concludes the work begun in Part I in present-
ing a coherent theory of conditioning consistent with all con-
ditional probability evaluations. Part I presented the interval
of events approach to conditional events, while Part II devel-
oped the cartesian product space approach. While the former is
computationally feasible to implement, it lacks certain theoret-
ical properties - in particular, it is non-boolean in nature.
On the other hand, the latter approach, while conceptually more
desirable than the former - it leads to a unique boolean struc-
ture - is much more complicated from a computation-implementation
viewpoint. This paper presents the more technically detailed
results required in the presentations of Parts I and II.

Key Words

OVERVIEW

This paper constitutes Part III of a series of three papers on the development of a theory of conditional information compatible with all conditional probability evaluations. The bulk of this paper is a mathematical appendix which supplies the technical details for the chief results provided in Parts I and II.

In Parts I and II of this paper, the problem of modeling conditional information compatible with all conditional probability evaluations was addressed [1], [2]. A typical problem is the determination of the meaning of the if-then statements "if b,then a" , "if d,then c", as well as the logical compound "if b,then a , and,if d, then c", and the probability P(if b,then a, and, if d, then c), when the probability evaluations of the component if-then statements are

$$P(\text{if } b, \text{ then } a) = P(a|b) \; ; \; P(\text{if } d, \text{ then } c) = P(c|d).$$

When b = d, the standard development of probability leads simply to

$$P(\text{if } b, \text{then } a, \text{ and, if } b \text{ then } c) = P(ac|b) = P(ab \cdot cb|b),$$

with the natural identifications

$$\text{"if } b, \text{ then } a\text{"} \leftrightarrow ab \; ; \text{"if } b, \text{ then } c\text{"} \leftrightarrow cb .$$

However, when b ≠ d , no standard procedure exists for dealing with the above and related issues. Call any such if-then statement as above a "conditional event" and the algebra that allows logically combining such entities a "conditional event algebra". In general, there are many such conditional event algebras, extending ordinary boolean algebra. (The term "extension" is used carefully here, since all unconditional events a,b,c,... can all be considered special cases of conditional events "if Ω, then a", "if Ω, then b", "if Ω, then c",..., where Ω is the universal event.) All of this leads, in turn, to the development of conditional events and conditional event algebras .

In Parts I and II, two types of conditional events were developed, useful in addressing the above type of problem as well as a wide variety of other problems that can arise in intelligent systems. The first type (Part I) has an event interval structure. Two leading calculi—or conditional event algebras determining the extension of the traditional boolean operations to this setting—are the GNW (Goodman-Nguyen-Walker) and SAC (Schay-Adams-Calabrese), which are briefly reviewed. The new type of conditional events(Part II - called α-type) arises

from an infinite product construction over the original sigma-algebra of unconditional events, and, unlike the first type, they form a proper subclass of the carrier space - which is here a sigma-algebra - and are , in general, not even closed relative to the boolean operations of the carrier space. But, a number of of problems that were not able to be addressed by the previous approach - including the higher order conditioning problem and the need for a firm basis for establishment of conditional random variables and related concepts - can be successfully addressed through the latter approach. However, one of the drawbacks in applying the new procedure is the rapidly increasing (factorially) computational lengths required for multiple argument conjunctions or disjunctions, unlike the essentially linear growth rate corresponding computations enjoyed by the interval approach. The usefulness of the two conditional event approaches was illustrated through the presentation and analysis of eight different types of conditional information problems.

Among the open issues deriving from this effort, the most important include:

1. Determine approximations to the product space approach operations which will retain accuracy, but will no longer be computationally intensive.

2. Complete characterization of optimal approximations to product space approach operations via GNW and SAC conditional event algebras.

3. Derive a universal property for the product space approach, justifying its form as opposed to other possible candidates.

4. Determine the structure of the subalgebra spanned by the second type (i.e., α-type) of conditional events, relative to the proper-including sigma algebra.

Returning to Part I, there is no question that Lewis' triviality result [3] was seen to be the basic guidepost that forced the search for conditional events and their algebras to be outside the basic boolean algebra (or sigma-algebra) of (unconditional) events a,b,c,d,... Section A here presents for completeness the Lewis theorem and its proof; Section B presents an alternative proof, based on probability ordering characterizations important also for their applications throughout Part I to the development of conditional event algebras.

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As stated before, the approach to conditional events taken in Part I is the interval of events one, where specifically any conditional event (a|b) (representing "if b, then a") is identified with

$$[ab, b' \vee ab] = \{x \in A : ab \leqslant x \leqslant b' \vee ab\}.$$

Section C here presents a rigorous argument for the above form to hold for conditional events. Although there are many possibilities for conditional event algebras, as mentioned earlier, the choices are not really arbitrary. In fact,

it can be shown that for a reasonably large class of conditional event algebras there is a bijective relation between each choice of a truth-functional three-valued logic and a conditional event algebra (relative to the interval of events approach). Section D of this appendix presents details of this relationship. It is also interesting to note that in general extensions of probability measures to conditonal event algebras do not produce probability measures over these algebras (but of course for each fixed antecedent, produce conditional probability measures). Section E provides details for this.

An alternate approach to conditional events and their algebras is provided by the cartesian product of spaces technique. Section F shows that as far as considering a product space approach, the (countable) infinite number of factors (of actually the same initial probability space) is necessary: no finite product space will yield conditional events. As mentioned previously, there is, in effect, a tradeoff between the theoretical soundness and computational efficiency of the two basic approaches to conditional events and their algebras. For the product space approach, Section G provides general formulas for conjunctions and their computational lengths and compares these quantitatively with the interval of events approach. Some progress on characterizing the product space approach to conditional events is given in Section H. Finally, Section I of this appendix shows how two of the leading conditional event algebras for the interval of events approach yield natural approximations (as upper and lower bounds in a tightest sense, for suitable modifications) to the full product space computations.

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MATHEMATICAL
APPENDIX

This appendix presents two types of results: concise statements of previously proven results, given usually without proof here (unless the latter illustrates a particular point) and full theorems and proofs for new results. The same convention for notation here holds as in Parts I and II.

A. Lewis' Triviality Result.

<u>Theorem</u> 1. (D. Lewis [3])

Let (Ω, A) be a measurable space. Suppose A contains at least some elements a,b such that

$$\emptyset < ab < b < \Omega . \qquad (1)$$

Then, it is impossible to find a mapping $\psi: A^2 \to A$ such that for all $a,b \in A$,

$$P(\psi(a,b)) = P(a|b) , \qquad (2)$$

for all probability measures $P: A \to [0,1]$ such that $P(b) > 0$.

<u>Proof</u>: Assume the converse and choose a fixed pair $a,b \in A$ with (1) holding such that

$$0 < P(ab) < P(b) < 1 \quad \& \quad P(ab) \neq P(a) \cdot P(b).$$

Then, by the assumption and usual properties of measurable spaces,

$$P(a|b) = P(\psi(a,b)) = P_a(\psi(a,b)) \cdot P(a) + P_{a'}(\psi(a,b)) \cdot P(a')$$

$$= P_a(a|b) \cdot P(a) + P_{a'}(a|b) \cdot P(a')$$

$$= 1 \cdot P(a) + 0 \cdot P(a')$$

$$= P(a) ,$$

a contradiction. ■

B. Probability Inequality and Equality Characterizations

Lewis' triviality result can also shown to be derivable as a direct consequent of the second of two basic theorems below concerning the ordering of conditional probabilities:

<u>Theorem</u> 2. ([4] , Lemma 2, pp, 48,49)

Let $a,b,c,d \in A$ with $b,d \neq \emptyset$ (A is a boolean or sigma-algebra, as usual). Then, the following two statements are equivalent:

(i) $\quad P(a|b) \leqslant P(c|d)$, all prob. meas. $P: A \to [0,1]$, $P(b), P(d) > 0$.

(ii) One of the following disjoint cases hold:

$\quad$ (I) $ab = \emptyset$,

in which case

$$P(a|b) = 0 \ , \ \text{all} \ P:A \rightarrow [0,1] \ ,$$

(II) $cd = d$,

in which case

$$P(c|d) = 1 \ , \ \text{all} \ P:A \rightarrow [0,1],$$

(III) $\emptyset < ab \leqslant cd \quad \& \quad \emptyset < c'd \leqslant a'b$.

Theorem 3. ([4], Corollary 1, p.49)

Let $a,b,c,d \in A$, $b,d \neq \emptyset$. The, the following two statements are equivalent:

(i) $P(a|b) = P(c|d)$, all $P:A \rightarrow [0,1]$, $P(b),P(d) > 0$.

(ii) One of the following disjoint cases holds:

(I) $ab = cd = \emptyset$,

in which case

$$P(a|b) = P(c|d) = 0 \ , \ \text{all} \ P:A \rightarrow [0,1] \ ,$$

(II) $ab = b \neq \emptyset \quad \& \quad cd = d \neq \emptyset$,

in which case

$$P(a|b) = P(c|d) = 1 \ , \ \text{all} \ P:A \rightarrow [0,1] \ ,$$

(III) $\emptyset < ab = cd < b = d$.

Alternative proof of Lewis' Theorem , using Theorem 3:

If eq.(2) holds in all P, then it can be rewritten as

$$P(\psi(a,b)|\Omega) = P(a|b) \ , \ \text{all} \ P:A \rightarrow [0,1] \ .$$

Immediately applying Theorem 3, shows the possibilities

(I) $\psi(a,b) = ab = \emptyset$

or

(II) $\psi(a,b) = \Omega \quad \& \quad ab = b$

or

(III) $\emptyset < \psi(a,b) = ab < \Omega = b$.

But, claerly, all of the above cases violate the hypothesis. Hence, a contradiction holds.

∎

C. Axiomatic Derivation of Interval Form of Conditional Events

Conditional Event Problem

Given measurable space (Ω, A) , find space $\tilde{A}$ with operations corresponding in some way to ordinary $\cdot, v, ()'$ over A (and with elements $\tilde{\emptyset}, \tilde{\Omega}$ corresponding to $\emptyset, \Omega$, respectively) and find a mapping $\psi: A^2 \to \tilde{A}$ such that

(Q1) $\psi(a,b) = \psi(ab,b)$, all $a,b \in A$.

(Q2) $\psi(\cdot, \Omega): A \to \tilde{A}$ is an injective isomorphism, i.e., an imbedding, with respect to $\cdot, v, ()'$.

(Q3) More generally than (Q2), for each $b \in A$, $b \neq \emptyset$, $\psi(\cdot, b): A \to \tilde{A}$ is a homomorphism.

(Q4) For each probability measure $P: A \to [0,1]$, there is a function $\tilde{P}: \tilde{A} \to [0,1]$ extending P in the sense

$$\tilde{P}(\psi(a,b)) = P(a|b) , \text{ all } a,b \in A , P(b) \geqslant 0$$

(Q5) For any $a,b,c,d \in A$ such that either $cd = \emptyset$ or $cd = d$, then

$$\psi(a,b) = \psi(c,d) \text{ implies } b = d .$$

Theorem 4. ([4], Chapter 2)

If the Conditional Event Problem has a solution relative to assuming properties (Q1)-(Q4), then for all $a,b,c,d \in A$, $b,d \neq \emptyset$,

(i) $\psi(a,b) = \psi(c,b)$ iff $ab = cb$.

(ii) $\psi(a,b) = \psi(c,d)$ iff Theorem 3 (ii) holds.

∎

Let (Ω, A) be a given measurable space and define the *natural mapping* nat: $A^2 \to \tilde{A}$, where for all $a,b \in A$,

$$nat(a,b) = Ab' \vee ab = \{xb' \vee ab : x \in A\} = [ab, b' \vee a] ,$$

etc. (For background on the role of the natural mapping, see any basic text in abstract algebra such as Burton [5], p.165 et passim.)

Define

$$\tilde{A} = range(nat) = \{nat(a,b) : a,b \in A\} ,$$

and note $nat(a,b) \in A/Ab'$, boolean quotient algebra generated by principal ideal $Ab' = \{xb' : x \in A\}$, with all of the usual coset operations. Hence,

$$\tilde{A} = \bigcup_{b \in A} A/Ab' = \bigcup_{c \in A} A/Ac ,$$

where for each b the coset operations are

$$nat(a,b) \cdot nat(c,b) = nat(ac,b) ,$$
$$nat(a,b) \vee nat(c,b) = nat(a \vee c,b),$$
$$nat(a,b)' = nat(a',b) .$$

Theorem 5.   ([4], chapter 2)

Let (Ω, A) be a measurable space. Then:

(i) $nat : A^2 \to \tilde{A}$ furnishes a solution to the Conditional Event Problem for properties (Q1)-(Q5), where, for any prob. meas. $P : A \to [0,1]$, $a,b \in A$, $P(b) > 0$,

$$\tilde{P}(nat(a,b)) = P(a|b) \text{ (by assignment or definition).}$$

(ii) For any space $\tilde{A}$ (not necessarily nat) and $\psi : A^2 \to \tilde{A}$ surjective satisfying at least properties (Q1),(Q3),(Q4),(Q5) of the Problem, then

$$\psi(a,b) = \psi(c,d) \text{ iff } ab = cd \ \& \ b = d   ,$$

and ψ is in a bijective relation with nat.

∎

Hence, without loss of generality, any ψ satisfying Theorem 5 is equivalent to nat, and we can now define conditional event "a given b" or "if b,then a" as $nat(a,b)$, where for purpose of brevity, we write $nat(a,b)$ simply as $(a|b)$ and define $(A|A) = \tilde{A}$.

D. Three-valued Logics and Conditional Event Algebras

Let (Ω, A) be a measurable space. We employ multivariable notation here:

$\underline{a} = (a_1,..,a_n) \in A^n$, where $a_j \in A$, $j=1,..,n$,

$(\underline{a}|\underline{b}) = ((a_1|b_1),..,(a_n|b_n))$; $\underline{a} \cdot \underline{b} = (a_1 b_1,..a_n b_n)$,

$\phi(\underline{a}|\underline{b})(\omega) = (\phi(a_1|b_1)(\omega),..,\phi(a_n|b_n)(\omega)) \in \{0,u,1\}^n$,

$\underline{i} = (i_1,..,i_n) \in \{0,u,1\}$,

$\phi(\underline{a}|\underline{b})^{-1}(\underline{i}) = (\phi(a_1|b_1)^{-1}(i_1),..,\phi(a_n|b_n)^{-1}(i_n))$,

$\cdot(\underline{a}) = a_1 \cdot a_2 \cdot .. \cdot a_n$ and similarly for $\cdot(\phi(\underline{a}|\underline{b})^{-1}(\underline{i}))$,

noting that
$$\phi(a_j|b_j)^{-1}(k) = \begin{cases} ab & , \text{ if } k=1, \\ a'b & , \text{ if } k=0, \\ b' & , \text{ if } k=u \end{cases},$$

using the three-valued indicator function form for conditional events $(a|b)$, $\phi(a|b): \Omega \to \{0,u,1\}$, where

$$\phi(a|b)(\omega) = \begin{cases} 1 & , \text{ if } \omega \in ab , \\ 0 & , \text{ if } \omega \in a'b, \\ u & , \text{ if } \omega \in b' \end{cases}.$$

It is easily seen that there is a bijection between all such three-valued indicator functions over Ω and $(A|A)$.

Also, call $f:(A|A)^n \to (A|A)$ a generalized boolean operation, if there are boolean operations $f_j: A^{2n} \to A$ such that

$$f(\underline{a}|\underline{b}) = (f_1(\underline{a} \cdot \underline{b}, \underline{b}) | f_2(\underline{a} \cdot \underline{b}, \underline{b})), \text{ for all } (\underline{a}|\underline{b}) \in (A|A)^n. \tag{3}$$

With all of the preliminaries completed, we can now state the main result:

Theorem 6. ([4], section 3.4)

(i) For any given generalized boolean operation $f:(A|A)^n \to (A|A)$, there is a unique corresponding three-valued truth-functional logical operation, say $\phi(f):\{0,u,1\}^n \to \{0,u,1\}$, such that ϕ is an isomorphism relative to f and $\phi(f)$, i.e.,

$$\phi(f(\underline{a},\underline{b}))(\omega) = \phi(f)(\phi(\underline{a}|\underline{b})(\omega)) , \text{ all } \omega \in \Omega .$$

Specifically, one can construct $\phi(f)$ as follows:
From eq.(3), using the normal disjunctive form modified, there is a unique

minimal index set $J(f_j) \subseteq \{0,u,1\}$ such that

$$f_j(\underline{a} \cdot \underline{b}, \underline{b}) = \bigvee_{\underline{i} \in J(f_j)} (\cdot(\phi(\underline{a}|\underline{b})^{-1}(\underline{i}))), \text{ all } \underline{a},\underline{b} \in A^n, j=1,2.$$

Then, define $\phi(f):\{0,u,1\}^n \to \{0,u,1\}$ by, for any $\underline{i} \in \{0,u,1\}^n$,

$$\phi(f)(\underline{i}) = \begin{cases} 1, & \text{if } \underline{i} \in J(f_1) \cap J(f_2), \\ 0, & \text{if } \underline{i} \in J(f_1)' \cap J(f_2), \\ u, & \text{if } \underline{i} \in J(f_2)'. \end{cases}$$

(ii) For any given three-valued truth-functional operation $\tau:\{0,u,1\}^n \to \{0,u,1\}$, there is a unique corresponding generalized boolean operation, say $\phi^{-1}(\tau):(A|A)^n \to (A|A)$, such that ϕ is an isomorphism relative to $\phi^{-1}(\tau)$ and τ, i.e.,

$$\tau(\phi(\underline{a}|\underline{b})(\omega)) = \phi(\phi^{-1}(\tau)(\underline{a}|\underline{b}))(\omega), \text{ all } \omega \in \Omega.$$

Specifically, one can construct $\phi^{-1}(\tau)$ as follows:
First, define as in eq.(3), the two components $(\phi^{-1}(\tau))_j$, $j=1,2$.

$$(\phi^{-1}(\tau))_j(\underline{a} \cdot \underline{b}, \underline{b})) = \bigvee_{\underline{i} \in R(h,j)} (\cdot(\phi^{-1}(\underline{a}|\underline{b})(\underline{i}))), \text{ all } \underline{a},\underline{b} \in A^n,$$

where
$$R(h,j) = \begin{cases} h^{-1}(1), & \text{if } j=1, \\ h^{-1}(\{0,1\}), & \text{if } j=2. \end{cases}$$

Finally, let

$$\phi^{-1}(\tau)(\underline{a}|\underline{b}) = ((\phi^{-1}(\tau))_1(\underline{a} \cdot \underline{b}, \underline{b}) | (\phi^{-1}(\tau))_2(\underline{a} \cdot \underline{b}, \underline{b})).$$

(iii)   The above results show that ϕ is an isomorphism between all truth-functionally-defined three-valued logics and all conditional event algebras ith operations being generalized boolean.

  ■

E. Non-Existence of Probability Measures Over the Conditional Event Extension of a Boolean Algebra

Theorem 7.
    Let (Ω,A) be a measurable space and pick $a,b,c,d \in A$ with $b,d \neq \emptyset$,

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$$S = \{0; \; x_i/(x_i+x_j), \; x_i \; ; \; x_i+x_j \; , \; \text{for } i,j \text{ distinct } \epsilon \; \{1,2,3\} \; ; 1\}.$$

But, by choosing, e.g.,

$$x_1 = x_2 = x_3 = 1/3 \quad ,$$

$$S = \{0,1/2,1/3,2/3,1\} \; ;$$

then, from above,

$$P(a|b) + P(c|d) = ((1/3)(2/3) + (1/3))/(2/3)$$

$$= 5/6$$

$$\notin S \; .$$

<u>Case</u> 3. $\quad bd \neq \emptyset \quad \& \quad abd = \emptyset$

Construct P so that
$$1/2 = P(ab) = P(b) = P(cd) = P(d) \; .$$
But, substituting into (2), if it were true, yields

$$P(a|b) + P(c|d) = 1+1 = 2 = P(e|f),$$

an impossibility.

<u>Case</u> 4. $\quad bd \neq \emptyset \quad \& \quad bcd = \emptyset$

Proof here imitates that for Case 3.

∎

<u>Corollary</u> 1.

Let (Ω, A) be a measurable space with eq.(1) satisfied above for some elements. Suppose also conditions (Q1)-(Q3) are satisfied relative to the Conditional Event Problem here. Suppose also that $\psi: A^2 \rightarrow \tilde{A}$ is closed with respect to conjunction and disjunction and that

$$\psi(a,b)\cdot\psi(c,d) = \tilde{\emptyset} \quad \text{implies} \quad abcd = \emptyset \; .$$

Then, even if $\tilde{A}$ were boolean, it is impossible for condition (Q4) to be satisfied when $\tilde{P}: \psi(A^2) \rightarrow [0,1]$ is an actual probability measure.

<u>Proof</u>: Immediate from Theorem 7.

∎

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F. No Finite Product Space Can Be Used To Model Conditional Events

Theorem 8.

Let A be a fixed nontrivial boolean algebra. Then, for any choice of $a, b \in$ A, with

$$\emptyset < a < b < \Omega \ , \tag{1}$$

there is no non-negative integer n and n order polynomial in four variables, $f_n : \mathbb{R}^4 \to \mathbb{R}$ with coefficients $c_{ijkt} \in \{-1, 0, 1\}$, $0 \leq i, j, k.t \leq n$, where, for any $x, y, z, w \in \mathbb{R}$

$$f_n(x,y,z,w) = \sum_{\substack{0 \leq i,j,k,t \\ i+j+k+t \leq n}} \left(c_{ijkt} \cdot x^i \cdot y^j \cdot z^k \cdot w^t \right)$$

is such that for all probability measures $P : A \to [0,1]$ with $P(a) > 0$,

$$f_n(P(a), P(b), P(a'b), P(b')) = P(a|b) \ .$$

Proof: Note first that since

$$P(a'b) = P(b) - P(a) \ ; \ P(b') = 1 - P(b) \ ,$$

the above problem is equivalent to having an n-order polynomial in two variables with coefficients $c_{ij} \in \{-1, 0, 1\}$, becoming now

$$g_n(x,y) = \sum_{i,j} c_{ij} x^i \cdot y^j \ ,$$

with, for all P,

$$g_n(P(a), P(b)) = P(a|b) \ . \tag{2}$$

Suppose that (2) is true for all P. In light of Lemma 0, [4], pp. 47-48, for the numbers 1/3 and 2/3, one can find P such that

$$P(a) = P(a'b) = P(b') = 1/3 \ ,$$

yielding

$$P(a|b) = (1/3)/((1/3)+(1/3)) = 1/2 \ .$$

Substituting these last two equations into (2) yields

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$$\sum_{i,j} c_{ij} \cdot (1/3)^i \cdot (2/3)^j = 1/2 \quad ,$$

which, when multiplied by $2 \cdot 3^{2n}$ yields

$$2 \cdot (\sum_{i,j} c_{ij} \cdot 3^{2n-i-j} \cdot 2^j) = 3^{2n}$$

which is impossible, as 3 to any power cannot have an even divisor. Hence, a contradiction occurs if (1) holds.

■

Corollary 2.

There is no finite sequence of finite-sized experiments consisting of independent trials whose outcomes are either a,a'b,b,b' (assuming as before that eq.(1) holds) such that the overall probability of success is $P(a|b)$.

Proof: Use Theorem 8.

■

G. Multiple Argument Form for Conjunction of α-Type Conditional Events and Their Computational Lengths

Theorem 9.

Let (Ω, A) be a given measurable space and construct as before $(\hat{\Omega}, \hat{A})$ with $\alpha: A^2 \to \hat{A}$, etc. Then, for all $a \leq b$, $c \leq d$, $e \leq f$, all in A :

(i)

$$\alpha(a,b) \cdot \alpha(c,d) \cdot \alpha(e,f)$$

mis ple

$$= \alpha(ad'f',bvdvf) \otimes \alpha(cf',dvf) \otimes \alpha(e,f)$$
$$v \; \alpha(acf',bvdvf) \otimes \alpha(e,f)$$
$$v \; \alpha(ad'f',bvdvf) \otimes \alpha(ce,dvf)$$
$$v \; \alpha(ad'f',bvdvf) \otimes \alpha(ed',dvf) \otimes \alpha(c,d)$$
$$v \; \alpha(aed',bvdvf) \otimes \alpha(c,d)$$
$$v \; \alpha(cb'f',bvdvf) \otimes \alpha(eb',bvf) \otimes \alpha(a,b)$$
$$v \; \alpha(ceb',bvdvf) \otimes \alpha(a,b)$$
$$v \; \alpha(cb'f',bvdvf) \otimes \alpha(ae,bvf)$$

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$\vee \; \alpha(cb'f',bvdvf) \otimes \alpha(af',bvf) \otimes \alpha(e,f)$

$\vee \; \alpha(eb'd',bvdvf) \otimes \alpha(ad',bvd) \otimes \alpha(c,d)$

$\vee \; \alpha(eb'd',bvdvf) \otimes \alpha(ac,bvd)$

$\vee \; \alpha(eb'd',bvdvf) \otimes \alpha(b'c,bvd) \otimes \alpha(a,b)$

$\vee \; \alpha(ace,bvdvf).$

(ii) For any probability measure $P:A \to [0,1]$, there is the corresponding product measure $\hat{P}:\hat{A} \to [0,1]$, yielding the evaluation of (i): $\hat{P}(\alpha(a,b) \cdot \alpha(c,d) \cdot \alpha(e,f))$ by replacing in (i) each occurrence of α by P, each comma within each original α-term by the conditional probability symbol $(\cdot | \cdot \cdot)$, $\otimes$ by arithmetic product, and $\vee$ by arithmetic sum.

Proof: Define for any $a,b \in A$,

$$b^j \times a = \begin{cases} a & , \text{ if } j=0 , \\ \underbrace{b \times \ldots \times b}_{j \text{ factors}} \times a & , \text{ if } j=1,2,\ldots . \end{cases}$$

Then,

$$\alpha(a,b) \cdot \alpha(c,d) \cdot \alpha(e,f)$$

$$= (\overset{+\infty}{\underset{i=0}{\vee}} (b')^i \times a) \cdot (\overset{+\infty}{\underset{j=0}{\vee}} (d')^j \times c) \cdot (\overset{+\infty}{\underset{k=0}{\vee}} (f')^k \times e)$$

$$= \underset{(\text{all } \rho)}{\vee} \; \underset{\substack{\text{all } i,j,k \\ \text{for same} \\ \text{rearrange-} \\ \text{ment } \rho \text{ rel-} \\ \text{ative to nu-} \\ \text{merical values,} \\ \text{counting ties}}}{\vee} (((b')^i \times a) \cdot ((d')^j \times c) \cdot ((f')^k \times e))) . \qquad (1)$$

By considering here all 13 possible rearrangements of i,j,k relative to increasing order, including cases for equalities, one obtains the desired result. For example, for the rearrangement $(j < i < k)$, which gives rise to the ninth term (from top to bottom) in the expansion in part (i), note that the corresponding term inside the right hand $\vee$ expression in eq.(1) becomes

$$(b'd'f')^j \times cb'f' \times (b'f')^{i-j-1} \times af' \times (f')^{k-i-1} \times e .$$

For (ii), the probability evaluation follows, again as in the two argument

case in Part II ([2], Section 4, main theorem, (vii)).

It is of some interest to be able to obtain the general formula for the conjunction of n α-type conditional events:

Let $\rho(\underline{j})$ be any rearrangement of $\underline{j} = (j_1,\ldots,j_n)$ in numerical order, taking into account ties

$$0 \leqslant j_{\rho(1,1)} = \cdots = j_{\rho(1,m_1)} < j_{\rho(2,1)} = \cdots = j_{\rho(2,m_2)} < \cdots < j_{\rho(r,1)} = \cdots = j_{\rho(r,m_r)} < +\infty,$$

so that

$$m_1 + m_2 + \cdots + m_r = n .$$

Also, for any index set K, let

$$v(a_K) = \underset{j \in K}{v} a_j \quad ; \quad \cdot(a_K) = \underset{j \in K}{\cdot} a_j ; \quad \cdot(a_K') = \underset{j \in K}{\cdot} a_j' ;$$

$$J_j = \{\rho(j,1),\ldots,\rho(j,m_j)\} , \quad j=1,\ldots,r ;$$

$$J^{(k)} = J_{k+1} \cup \cdots \cup J_r , \quad k=0,1,\ldots,r-1 ; \quad J^{(0)} = J_1 \cup \cdots \cup J_r = \{j_1,\ldots,j_n\},$$

$$J^{(r-1)} = J_r ; \quad J^{(r)} = \emptyset .$$

Then, for any $a_j \leqslant b_j \in A$,

$$\cdot(\alpha(\underline{a},\underline{b})) = \alpha(a_1,b_1) \cdot \ldots \cdot \alpha(a_n,b_n) \quad ,$$

$$\alpha(a_i,b_i) = \overset{+\infty}{\underset{j_i=0}{v}} (b_i')^{j_i} \times a_i \quad . \tag{2}$$

It follows analogous to the case of three arguments,

$$\cdot(\alpha(\underline{a},\underline{b})) = \underset{\text{all } \rho}{v} \underset{\left(\begin{array}{c}\text{all } \underline{j}, \\ \text{for } \rho \\ \text{fixed}\end{array}\right)}{v} \gamma(\rho(\underline{j})) \quad ,$$

where

$$\gamma(\rho(\underline{j})) = (\cdot(b_{J(0)}'))^{j_{\rho(1,1)}} \times (\cdot(a_{J_1}) \cdot (\cdot(b_{J(1)}')))$$

$$\times (\cdot(b_{J(1)}'))^{j_{\rho(2,1)} - j_{\rho(1,1)} - 1} \times (\cdot(a_{J_2}) \cdot (\cdot(b_{J(2)}')))$$

$$\times \cdots$$

$$\times (\cdot(b_{J(r-2)}'))^{j_{\rho(r-1,1)} - j_{\rho(r-2,1)} - 1} \times (\cdot(a_{J_{r-1}}) \cdot (\cdot b_{J(r-1)}'))$$

$$\times (\cdot(b_{J(r-1)}'))^{j_{\rho(r,1)} - j_{\rho(r-1,1)} - 1} \times (\cdot(a_{J_r})) . \tag{3}$$

Hence, taking disjunctions over the $J_{\rho(i,1)}$ in (3), from basic representation for each $\alpha(a_i, b_i)$ in (2), it follows that:

Theorem 10.

Let the same assumptions hold as above. Then,
(i)

$$\cdot(\alpha(\underline{a}, \underline{b})) = \underset{\substack{\text{all} \\ \rho}}{v} \ \gamma(\rho) \ ,$$

where for any rearrangement ρ, noting the dependency of the $J^{(j)}$ and J_i and r on ρ ,

$$\gamma(\rho) = \underset{i=1}{\overset{r}{\otimes}} \alpha(\cdot(a_{J_i}) \cdot (\cdot(b'_{J(i)})) , \ v(b_{J(i-1)})) \ ,$$

indicating repetitive tensor-like producting in the obvious way, noting its reduction to an identity form when $r=1$.

(ii) For any probability evaluation of (i) via the product measure extension $\hat{P}$ of P , $\hat{P}(\cdot(\alpha(\underline{a}, \underline{b}))$ is obtained , analogous to Theorem 9(ii), by replacing in (i) each occurrence of α by P, each comma inside the original α-type conditional event by conditional probability operation $(\cdot|\cdot\cdot)$, $\otimes$ by arithmetic product, and v by arithmetic sum. ∎

Next, consider the computational lengths required to obtain repeated α-type conditional event conjunction and disjunction. First, recall (see, e.g., Abramowitz & Stegun [6], section 24) the total number of ways to partition a set of n distinct elements into k non-empty components is the Stirling number of the second kind

$$S_n^{(k)} = (1/k!) \sum_{j=0}^{k} (-1)^{k-j} \cdot \binom{k}{j} \cdot j^n \ ,$$

and the number $\kappa(n)$ of possible distinct rearrangements ρ of n distinct variable set $\{j_1, \ldots, j_n\}$,where ties are allowed and the rearrangements are relative to increasing numerical order, is

$$\kappa(n) = \sum_{k=1}^{n} S_n^{(k)} \cdot k! \ .$$

Hence,

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$$\kappa(n) = \sum_{k=1}^{n} \sum_{j=0}^{k} (-1)^{k-j} \cdot \binom{k}{j} \cdot j^n \ . \tag{4}$$

Using the above results, one can compare the computational lengths between the two leading conjunction operation candidates (GNW and SAC) for the interval representation form of conditional events with the computational length for the α-representation of conditional events conjunction:

Theorem 11.

Let (Ω, A) be a given measurable space. Consider both its interval conditional event extension $(A|A)$ relative to GNW and SAC algebras. Consider also the product space extension $(\hat{\Omega}, \hat{A})$ and the associated α-type conditional event extension $\alpha(A^2) \subset \hat{A}$. Then, for any $a_j \leqslant b_j \in A$, $j=1,..,n$:

(i) Number of boolean operations required to compute $P(\cdot((\underline{a}|\underline{b})))$ for GNW is A_n , where

$$A_n = 5n \ .$$

(ii) Number of boolean operations required to compute $P(\cdot((\underline{a}|\underline{b})))$ for SAC is B_n , where

$$B_n = 4n \ .$$

(iii) Number of boolean operations required to compute $P(\cdot(\alpha(\underline{a},\underline{b})))$ is C_n , where

$$\kappa(n) \leqslant C_n \leqslant 2n^2 \kappa(n) \ .$$

Proof: For (i), use the form

$$\cdot((\underline{a}|\underline{b})) = (\cdot(\underline{a})|(\cdot(\underline{a})) \vee v(\underline{a}' \cdot \underline{b}))$$

and for (ii), use the form

$$\cdot_0((\underline{a}|\underline{b}) = (\cdot(\underline{b}' \vee \underline{a}) | v(\underline{b})),$$

where $\cdot$ is GNW conjunction and $\cdot_0$ is SAC conjunction. See Section 3(iii) [1], eq.(1) for the two-argument case for GNW and Remark 2, following eq.(1) above for the two-argument case for SAC. Both generalize readily to the multiple argument forms.

H. Uniqueness of Representation of α-Type Conditional Events

The closest we can now come to establishing the uniqueness of the α-representation for conditional events is in the following theorem:

Theorem 12.

Given a measurable space (Ω, A) and infinite product space extension $(\hat{\Omega}, \hat{A})$ with conditional event mapping $\alpha: A^2 \to \hat{A}$; consider

$$A_r = \sigma(\{a_1 \times .. \times a_r : a_j \in A, \ j=1,..,r\}).$$

Let $A, B: A^2 \to A_r$ be arbitrary satisfying

$$\hat{P}(A(a,b) | B(a,b)) = P(a|b) \ , \text{ all } a,b \in A, \text{ all } P \ .$$

Suppose, also that $\psi: A^2 \to \hat{A}$ is such that

$$\psi(a,b) = \overset{+\infty}{\underset{i=0}{\dot{v}}} \ \overset{+\infty}{\underset{j=0}{\times}} \ c_{ij} \ , \tag{0}$$

where $c_{ij} = c_{ij}(A(a,b), B(a,b)) \in A_r$, for all $a,b \in A$, all $i,j=0,1,2,..$.

and $\dot{v}$ indicates disjoint disjunction.

Then, the following two statements are equivalent:

(i)
$$c_{ij} \in \{A(a,b) \cdot B(a,b), B(a,b)', \Omega^r\} \ , \text{ all } i,j=0,1,2,... \tag{1}$$

and
$$\hat{P}(\psi(a,b)) = P(a|b) \ , \text{ all } a,b \in A , \text{ all } P \ . \tag{2}$$

etc.

(ii) Up to relative factor space locations, ψ is the same formally as the canonical expansion for $\alpha(a,b)$ (Part II, Section 4, eq.(6)), with a replaced by $A(a,b)$ and b by $B(a,b)$

$$\psi(a,b) = (A(a,b) \cdot B(a,b)) \ v \ (B(a,b)' \times A(a,b) \cdot B(a,b))$$
$$v \ (B(a,b)' \times B(a,b)' \times A(a,b) \cdot B(a,b)) \ v.. \tag{3}$$

More precisely, there is (uniquely determined by each choice of c_{ij}'s) a sequence $\underline{s} = (s_k)_{k=0,1,2,..}$ of distinct non-negative integers such that

$$c_{ij} = \begin{cases} A(a,b) \cdot B(a,b) & \text{, if } j=s_i \text{ ,} \\ B(a,b)' & \text{, if } j=s_0,s_1,\ldots,s_{i-1} \text{ ,} \\ \Omega^r & \text{, if } j \neq s_0,s_1,\ldots,s_{i-1},s_i \text{ ,} \end{cases} \quad (4)$$

for all $i,j = 0,1,2,\ldots$

Proof: Given (ii), one clearly has the same situation holding as in the canonical one given in (3), up to a fixed permutation of the relative locations of $A(a,b) \cdot B(a,b)$ and $B(a,b)'$. Thus it follows (Part II, eq.(6)) with a replaced by $A(a,b)$, b by $B(a,b)$, that (2) holds. (That ((4) implies (1) is obvious.). Hence, (i) holds.

Given (i) holding, use (1) ,(2) and the expansion (0) with $x=P(A(a,b) \cdot B(a,b))$, $y=P(B(a,b)')$ replacing $P(ab)$, $P(b')$, respectively:

$$P(a|b) = P(\psi(a,b)) = \sum_{i=0}^{+\infty} \prod_{j=0}^{+\infty} \hat{P}(c_{ij}) = \sum_{i=0}^{+\infty} x^{t_i} \cdot y^{w_i}$$

$$= \hat{P}(A(a,b)|B(a,b)) = \sum_{i=0}^{+\infty} x \cdot y^i \text{ ,} \quad (5)$$

for all $a,b \in A$, all P, for some non-negative integers t_i , w_i. Thus, x and y can vary freely over some domain and hence the equality of the two power series in (5) implies, without loss of generality,

$$t_i = 1 \text{ , } w_i = i \text{ , for } i=0,1,2,\ldots \text{ .} \quad (6)$$

In turn, (6) implies, by first considering i=0, then i=1, then i=2,..., using the mutual orthogonality of each disjunction term, that (4) indeed holds for some $\underline{s}$, and hence (ii) holds. ∎

One can add constraints to the assumptions of the last theorem and to either equivalent form (i) or (ii) to eliminate all $A(a,b)$ and $B(a,b)$ except for the simplest: $A(a,b)=a$, $B(a,b)=b$. For example, if one assumes the modus ponens relation

$$\psi(a,b) \cdot b = ab \text{ , all } a,b \in A$$

and, e.g.,

$$ab \leqslant B(a,b) \text{ , all } a,b \in A \text{ ,}$$

then it easily is shown that a,b are the smallest possible A,B candidates:

$$ab \leqslant A(a,b) \cdot B(a,b) \text{ , } b \leqslant B(a,b), \text{ all } a,b \in A \text{ .}$$

Other than the above result, little progress has been made in fully characterizing α-type representation. Of course, if one could show, e.g., that assumptions led to the conclusion that candidate conditional event form, say $\mu(a,b)$ satisfied the modus ponens condition in (1) (with μ for γ) and, as well

$$\mu(a,b) \cdot b' = b' \times \mu(a,b) \text{ ,} \tag{2}$$

a condition that is compatible always with the independence relation

$$\hat{P}(\mu(a,b) \cdot b') = \hat{P}(\mu(a,b)) \cdot \hat{P}(b') = P(a|b) \cdot P(b') \text{ , all prob. } P,$$

then, by the simple expansion

$$\mu(a,b) = \mu(a,b) \cdot b \vee \mu(a,b) \cdot b' \tag{3}$$

and reiteratively using (2) and (3), one obtains the infinite sequence expansion of $\alpha(a,b)$.

I. Approximation of α-Type Operations by GNW and SAC Operations

First, consider a basic comparison between GNW, SAC, and α-type operations corresponding to $\cdot$, $\vee$, $()'$, where (Ω,A) is a given measurable space, $(\hat{\Omega},\hat{A})$ is its infinite product extension, and $\alpha : A^2 \to \hat{A}$, etc. In all of the following, as before, we use the convention that $\cdot_0$, $\vee_0$ refer to SAC conjunction, disjunction, respectively , and now, $\cdot_1$, $\vee_1$ refer to GNW conjunction, disjunction, respectively, with unsubcripted $\cdot$, $\vee$ referring to α-type conjunction, disjunction, respectively. Finally, $()'$ refers to either the common GNW and SAC interpretation or the α-type interpretation of complement/negation.

Theorem 13.

For any $a,b,c,d \in A$ leading to nontrivial $\alpha(a,b)$ (i.e., $ab \neq \emptyset$, $b \neq \emptyset$, so $\alpha(a,b) \neq \hat{\emptyset}$ and $ab \neq b$, $b \neq \emptyset$, so that also $\alpha(a,b) \neq \hat{\Omega}$):

(i)

$$\alpha(a,b)\cdot_1 \alpha(c,d) \leqslant \alpha(a,b)\cdot\alpha(c,d) \leqslant \alpha(a,b)\cdot_0 \alpha(c,d) \ ,$$

$$\alpha(a,b)v_0 \alpha(c,d) \leqslant \alpha(a,b)v\alpha(c,d) \leqslant \alpha(a,b)v_1 \alpha(c,d) \ ,$$

where the GNW and SAC analogues for α-type conditionals are:

$$\alpha(a,b)\cdot_1 \alpha(c,d) = \alpha(abcd , a'b \ v \ c'd \ v \ bd) \ ,$$

$$\alpha(a,b)v_1 \alpha(c,d) = \alpha(ab \ v \ cd , ab \ v \ cd \ v \ bd)'$$

$$\alpha(a,b)\cdot_0 \alpha(c,d) = \alpha(abd' \ v \ cdb' \ v \ abcd , b \ v \ d) \ ,$$

$$\alpha(a,b)v_0 \alpha(c,d) = \alpha(ab \ v \ cd , b \ v \ d) \ .$$

(1)

(ii) The bounds by the GNW operations given as above are the tightest possible, i.e., referring to Remark 3, following the main theorem, Section 4, for all nontrivial α-type conditional events

$$\alpha(a,b)\cdot_* \alpha(c,d) = \alpha(a,b)\cdot_1 \alpha(c,d) \ ,$$

$$\alpha(a,b)v^* \alpha(c,d) = \alpha(a,b)v_1 \alpha(c,d) \ .$$

Proof: First, note from Section 4, main theorem, part (iv), the partial (lattice) order $\leqslant$ for $(A|A)$ for GNW coincides with the (boolean) order stemming from $\hat{A}$ restricted to the space of α-type conditionals, $\alpha(A^2)$. This implies directly that the upper left inequality is true. Dually, because GNW and α-type conditionals both form DeMorgan systems for conjunction, disjunction, and negation, reapplication of the above result then shows the validity of the lower right inequality.

The upper right inequality holds by inspection of the conjunction form for SAC and that for α-type, showing the difference to be in the appended tensor-like factors to the latter. (These factors are readily seen to yield expressions $\leqslant$ corresponding SAC terms without the tensor factors.) Finally, since both SAC and α-type conjunction , disjunction, and negation form a DeMorgan system, the lower left inequality is also valid.

■

Modify the definitions of $\cdot^*$ and v_* from Remark 3, following main theorem of Section 4, as follows:

$$\alpha(a,b) \cdot \underset{(\leqslant)}{\textcircled{$\ominus$}} \alpha(c,d) = \inf \{\alpha(e,f): \alpha(a,b) \cdot \alpha(c,d) \leqslant \alpha(e,f), \text{ for which Hypothesis(o)}$$
$$\text{is satisfied} \} \quad ,$$
$$\alpha(a,b) \underset{(\leqslant)}{\textcircled{$\oplus$}} \alpha(c,d) = \sup \{\alpha(e,f): \alpha(e,f) \leqslant \alpha(a,b) \vee \alpha(c,d), \text{for which Hypothesis(o)}$$
$$\text{is satisfied} \}$$

<u>Hypothesis(o)</u>: For any $a,b,c,d \in A$, consider only $\alpha(a,b), \alpha(c,d) \neq \hat{\emptyset}, \hat{\Omega}$, with

$b \cdot (d' \vee c)$, $d \cdot (b' \vee a) \neq \emptyset$; $ef \cdot (d' \vee c) \neq f \cdot (d' \vee c)$; $ef \cdot (b' \vee a) \neq f \cdot (b' \vee a)$.

<u>Theorem 14.</u>

For any $a,b,c,d \in A$, the above modified upper optimal approximation for conjunction and lower approximation for disjunction coincide with the corresponding SAC operations. That is, for all such (restricted) a,b,c,d ,

and
$$\alpha(a,b) \textcircled{$\ominus$} \alpha(c,d) = \alpha(a,b) \cdot_o \alpha(c,d)$$
$$\alpha(a,b) \textcircled{$\oplus$} \alpha(c,d) = \alpha(a,b) \vee_o \alpha(c,d).$$

(2)

<u>Proof</u>: Let $e,f \in A$ arbitrary such that

$$\alpha(a,b) \cdot \alpha(c,d) \leqslant \alpha(e,f) . \tag{3}$$

Using Hypothesis (o), let $P: A \to [0,1]$ be any probability measure with $P(b \cdot (d' \vee c)) > 0$ and denote, as usual, the infinite product probability associated with P as $\hat{P}: \hat{A} \to [0,1]$. We then apply the conditional probability $\hat{P}_{b \cdot (d' \vee c)}$ to both sides of eq.(3), yielding (after simplifying the LHS of (3) (see the main theorem, part(vii), Section 4))

$$P(abd' \vee abcd \mid b \cdot (d' \vee c)) \leqslant P(e \mid f \cdot (d' \vee c)) , \tag{4}$$

for all prob. meas. P. Hence, by Theorem 2, Appendix and Hypothesis (o), the only possibility is that eq.(4) implies

$$abd' \vee abcd \leqslant ef \cdot (d' \vee c) \leqslant ef . \tag{5}$$

Similarly, by taking now $P_{d \cdot (b' \vee a)}$ and applying its infinite product extension to both sides of (3) shows, analogous to (4),

$$cdb' \vee abcd \leqslant ef \cdot (b' \vee a) \leqslant ef . \tag{6}$$

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Then, disjoining eqs.(5) and (6) gives

$$abd' \lor cdb' \lor abcd \leqslant ef . \qquad (7)$$

Also, by eq.(3) and Theorem 13(i), it follows that

$$\alpha(a,b) \neg \alpha(c,d) \leqslant \alpha(e,f) ,$$

whence by Section 3(iii), eq.(4),

$$abcd \leqslant ef \ \& \ e'f \leqslant (abcd)'(a'b \lor c'd \lor bd) = a'b \lor c'd. \qquad (8)$$

Combining eqs.(7) and (8),

$$abd' \lor cdb' \lor abcd \leqslant ef \ \& \ e'f \leqslant a'b \lor c'd . \qquad (9)$$

On the other hand, consider the comparison of SAC conjunction with $\alpha(e,f)$: By eq.(1) above and again use of eq.(4), Section 3(iii),

$$\alpha(a,b) \cdot_0 \alpha(c,d) \leqslant \alpha(e,f) \qquad (10)$$

iff

$$abd' \lor cdb' \lor abcd \leqslant ef \ \& \ e'f \leqslant (abd' \lor cdb' \lor abcd)' \cdot (b \lor d)$$
$$= a'b \lor c'd . \qquad (11)$$

But, clearly, eqs.(9) and (11) are the same. Hence, eq.(10) holds. Since $\alpha(e,f)$ is also arbitrary satisfying eq.(3), then by the very definition for $\odot$, the top equation of (2) holds. By a duality argument (using DeMorgan relations) the bottom equation of (2) also holds.

■

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